Quantum-Corrected Entropy for 1+1-Dimensional Gravity Revisited

by

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ABSTRACT

In this paper, we examine a generic theory of 1+1-dimensional gravity with coupling to a scalar field. Special attention is paid to a class of models that have a power-law form of dilaton potential and can capably admit black hole solutions. The study focuses on the formulation of a Lorentzian partition function. We incorporate the principles of Hamiltonian thermodynamics, as well as black hole spectroscopy, and find that the partition function can be expressed in a well-defined, calculable form. We then go on to extract the black hole entropy, including the leading-order quantum correction. As anticipated, this correction can be expressed as the logarithm of the classical entropy. Interestingly, the prefactor for this logarithmic correction disagrees, in both magnitude and sign, with the findings from a prior study (on the very same model). We comment on this discrepancy and provide a possible rationalization.

1 Introduction

There is a long and storied history to the branch of gravitational physics known as black hole thermodynamics. It began in the mid-seventies, when Bekenstein [1] and Hawking [2] formulated a convincing analogy between black hole mechanics and the laws that govern a thermodynamic system. Hawking dramatically punctuated this analogy with his demonstration that black holes emit thermal radiation (via a quantum tunneling mechanism) at a temperature that complies perfectly with the value predicted by the first law [3].

There has since been an "endless" series of research papers that have examined black hole thermodynamics from a myriad of different angles [4]. The motivation to study black holes has stemmed from several sources; for instance, their likely existence as astronomical objects, the role of microscopic holes in the primordial universe, as well as simple aesthetic interest. However, the most compelling motivating factor might be in the context of quantum gravity, where black holes have served as a particularly useful testing ground for theoretical frameworks and ideas. In this regard, black hole entropy occupies a very esteemed position, as it is often argued that any viable theory of quantum gravity must naturally incorporate and ultimately explain the Bekenstein-Hawking area law [1, 2].

This important role of black hole entropy also serves to magnify the conceptual difficulties that it unfortunately presents. Most significantly, the information loss paradox [5], the breakdown of quantum field theory (or, equivalently, the holographic storage of information [6]), and the statistical origin of the entropy in question [7]. That is not to say there has been a lack of progress in answering these questions; if anything, there has been too much! For instance, various microstate-counting frameworks have successfully reproduced the Bekenstein-Hawking entropy; including those with their origins in string theory [8], quantum geometry [9], Sakharov's induced gravity [10], Chern-Simons theory [11], asymptotic conformal symmetries [12], *etcetera*. It is, however, conspicuously unclear as to how these various techniques are related and, moreover, as to what degrees of freedom are truly being counted. Without some fundamental theory that ties this altogether, the statistical origin of the entropy remains as enigmatic as ever.

It should be evident that we need to better our understanding of black hole entropy; however, it remains quite unclear as to how substantial progress can be made. One possible direction, which has enjoyed success in the past, is to investigate theoretical models of two-dimensional gravity [13]. Such models allow one to focus directly on the essence of the physical issues, without being bogged down by the calculational complexities inherent to higher-dimensional theories.

Although the physical significance of two-dimensional gravity can, in general, be challenged, there are many specific instances where a given model is directly related to a physically relevant theory. As an example, let us consider Jackiw-Teitelboim theory [14], which describes constant-curvature black hole solutions in two-dimensional anti-de Sitter space. This model happens to have a dual relationship with certain string-inspired black holes [15] and the near-extremal sector of the Reissner-Nordstrom black hole [16]. Moreover, the Jackiw-Teitelboim model can also be viewed as a dimensionally reduced form of the always topical BTZ black hole [17, 18].

With the above discussion in mind, the current paper considers the black hole thermodynamics of a general model of 1+1-dimensional gravity coupled to a dilatonic scalar field. One of the principal achievements will be the formulation of a Lorentzian partition function, which is inspired by the Hamiltonian thermodynamic analysis of Kuchar [19] and others (for instance, [20, 21, 22]). To express the partition function in a suitable form for explicit analysis, we also call upon some other intriguing facets of black hole quantum theory; including spectroscopy [23], complementarity [24] and spacetime Euclideanization [25]. In this sense, our formalism provides a simplified (lower-dimensional) framework for illustrating some of the essentials of black hole physics.

Our ultimate objective will be to evaluate the black hole entropy, including the leading-order quantum correction to the classical thermodynamic value. In this regard, it is interesting to take note of a related work by the current author [26]. This recent paper considered the quantum-corrected entropy for precisely the same two-dimensional model. However, the approach of [26] was very much different from that of the present treatment. In fact, [26] was based on two distinct analytic methods for calculating the entropy; with these being as follows:

Firstly, we employed a formula that was derived by Das *et al* [27] and follows from purely thermodynamic principles. Secondly, we applied Carlip's quantum-corrected version [28] of the renowned Cardy formula [29].[1]

For both of these prior methods, the leading-order correction was shown to be proportional to the logarithm of the classical entropy. This outcome is in agreement with various calculations that are found in the literature; for instance, in the quantum-geometry analysis of Kaul and Majumdar [31]. (Also see [32]-[36],[28],[37]-[40],[27].) However, there was also a notable discrepancy between the two methods of [26]: the statistical (i.e., Cardy) approach consistently yields a logarithmic prefactor of $-3/2$, whereas the thermodynamic treatment generates a model-dependent prefactor. Interestingly, the two prefactors are only in agreement for the very special case of Jackiw-Teitelboim theory [14].

It should be noted that a logarithmic prefactor of $-3/2$ coincides precisely with the value prescribed by quantum geometry [31]. This particular value turns up elsewhere in the literature, although not exclusively so. For instance, a four-dimensional study that is close in spirit to the current two-dimensional analysis found a logarithmic prefactor of $+1/2$ [35]. One might anticipate that the same outcome will be found here; but, as always in physics, there can be no substitute for an explicit check.

The remainder of the paper is organized as follows. In Section 2, we introduce a generic model of 1+1-dimensional gravity with coupling to a scalar field. We then go on to discuss the black hole solutions and associated thermodynamics at the classical level. In Section 3, we consider a Lorentzian partition function for the 1+1-dimensional black hole of interest. With the help of Hamiltonian thermodynamics and black hole spectroscopy, we are able to express the partition function in a convenient form for explicit analysis. In Section 4, we use the prior derivation as the basis for a calculation of the black hole entropy. With the assumption of a semi-classical regime, we are able to express the entropy in terms of a well-defined perturbative expansion. Section 5 contains a summary and overview. In particular, we directly compare the current results with the findings from our prior study.

[1]In using the Cardy-Carlip formula, we first demonstrated that the near-horizon geometry of the generic theory can be effectively described by a conformal field theory. Moreover, this conformal theory was shown to have an identifiable Virasoro algebra. This portion of the analysis generalized a treatment by Solodukhin [30].

2 Generic Dilaton Gravity

Let us begin by considering a 1+1-dimensional theory of gravity coupled to a dilaton (or auxiliary scalar) field.[2] Assuming a diffeomorphism invariant action that contains at most second derivatives of the fields, we have:

$$I = \frac{1}{2G}\int d^2x\sqrt{-\overline{g}}\left[D(\overline{\phi})R(\overline{g}) + \frac{1}{2}\overline{g}^{\mu\nu}\nabla_\mu\overline{\phi}\nabla_\nu\overline{\phi} + \frac{1}{l^2}U(\overline{\phi})\right]. \tag{1}$$

Here, G is a dimensionless measure of gravitational coupling (i.e., the two-dimensional "Newton constant"), l is a fundamental constant of dimension length, while $D(\overline{\phi})$ and $U(\overline{\phi})$ are well-behaved but otherwise arbitrary functions of the dilaton field. It is a point of interest that the very same action can also describe dilaton-gravity coupled to an Abelian gauge field.[3]

The viewpoint of this paper will be that Eq.(1) represents some sort of fundamental theory. Nevertheless, it is worth noting the above action can often have physical significance from the perspective of higher-dimensional theories. For instance, one finds that $D = \overline{\phi}^2/4$ and $U = 1$ after the spherical reduction of 3+1-dimensional Einstein gravity [42]. Furthermore, many two-dimensional theories have near-horizon dualities with either near-extremal black holes [16] or string-theoretical models [43].

Exploiting conformal symmetry, one can conveniently reformulate the action so that the kinetic term is eliminated. This task can, in fact, be accomplished by way of the following field redefinitions [44]:

$$\phi = D(\overline{\phi}), \tag{2}$$

$$g_{\mu\nu} = \Omega^2(\overline{\phi})\overline{g}_{\mu\nu}, \tag{3}$$

$$\Omega^2(\overline{\phi}) = \exp\left[\frac{1}{2}\int^{\overline{\phi}} d\xi \left(\frac{dD}{d\xi}\right)^{-1}\right], \tag{4}$$

$$V(\phi) = \frac{U(\overline{\phi})}{\Omega^2(\overline{\phi})}. \tag{5}$$

[2] The dilaton is a necessary element, inasmuch as the Einstein tensor identically vanishes for two-dimensional gravity.

[3] Assuming a gauge-invariant action, one finds that the Abelian sector can be exactly solved in terms of the dilaton and a conserved charge [41]. Hence, the total action can always be re-expressed in the form of Eq.(1).

In the above usage, it is implicit that $D(\overline{\phi})$ and its derivative are non-vanishing throughout the relevant manifold.

In its reparametrized form, the action (1) simplifies as follows:

$$I = \frac{1}{2G}\int d^2x\sqrt{-g}\left[\phi R(g) + \frac{1}{l^2}V(\phi)\right]. \tag{6}$$

The field equations of this reparametrized action can readily be solved. Moreover, for the submanifold $x \geq 0$, this solution can be conveniently expressed in a static, "Schwarzschild-like" gauge [45]:

$$\phi = \phi(x) = \frac{x}{l} \quad \geq 0, \tag{7}$$

$$ds^2 = -\left(J(\phi) - 2lGM\right)dt^2 + \left(J(\phi) - 2lGM\right)^{-1}dx^2, \tag{8}$$

$$J(\phi) = \int^{\phi} d\xi V(\xi), \tag{9}$$

where M is a constant of integration. We will regard M as being non-negative; in which case, it can be identified with the conserved mass of a black hole solution (if such a solution exists).

In the analysis to follow, we will assume that the theory always admits black hole solutions for which the outermost horizon, $\phi_o = x_o/l$, is non-degenerate.[4] In very general terms, a black hole horizon can be identified with a hypersurface of vanishing Killing vector [47]. For the model under consideration, this identification translates into the following relation [48, 45]:

$$J(\phi_o) - 2lGM = 0. \tag{10}$$

Alternatively, one can write:

$$\phi_o = J^{-1}(2lGM). \tag{11}$$

Next, let us consider the tree-level black hole thermodynamics. It is pertinent to this discussion that all thermodynamic properties of current interest are invariant under the above reparametrization [49]. Firstly, we can calculate the Hawking temperature (T_o) via the usual Euclidean prescription

[4]It is sufficient criteria for a black hole solution if $J(\phi) - 2lGM$ has a zero and $\frac{dJ}{d\phi} > 0$ in the relevant manifold ($\phi \geq 0$). For a more rigorous discussion, see [46].

[25]; that is, the inverse temperature is identified with the periodicity of imaginary time. One readily finds that:

$$T_o = \frac{1}{4\pi l} \left. \frac{dJ}{d\phi} \right|_{\phi_o} . \tag{12}$$

Given a two-dimensional spacetime, there is no obvious way of defining the area of a one-dimensional surface, such as the horizon area of a black hole. For this reason, the Bekenstein-Hawking area law [1, 2] can not be exploited in a straightforward manner. Nonetheless, we can still ascertain the thermodynamic entropy (S_o) by virtue of the first law of thermodynamics: $dM|_{horizon} = T_o dS_o$. Integration of this expression yields:

$$S_o = \frac{2\pi}{G}\phi_o, \tag{13}$$

where the arbitrary constant has been set to zero in accordance with the usual convention. Working backwards from this result, we can now define an effective horizon "area" as follows:

$$A_o = 4GS_o = 8\pi\phi_o. \tag{14}$$

Here, we have simply applied the standard Bekenstein-Hawking definition.

So far, we have treated the dilaton potential, $U(\overline{\phi})$ or $V(\phi)$, as generically as possible. However, for illustrative purposes, it will often prove to be instructive and convenient if this potential is given an explicit form. Following our prior, related work [26], we thus consider a "power-law" potential:

$$V(\phi) = \gamma\phi^a, \tag{15}$$

where a and γ are dimensionless, non-negative, model-dependent parameters. Interestingly, this form of potential can describe a Weyl-rescaled CGHS model (for $a = 0$ and $\gamma = 1$) [50, 51], or a dimensionally reduced BTZ black hole (for $a = 1$ and $\gamma = 2$) [17, 18]. On a more general note, such a potential can capably describe (after suitable rescalings) the near-horizon geometry of a single-charged dilatonic black hole, a multi-charged stringy black hole, or a dilatonic p-brane [43]. Also, if a is allowed to be less than zero, than this potential can describe the spherical reduction of d-dimensional Einstein gravity (for $a = -1/(d-2)$ and $\gamma = (d-3)/(d-2)$ [22]). A negative value of

a, however, somewhat complicates the later analysis.[5] Hence, we will generally enforce $a \geq 0$ and comment further on this restriction at an appropriate point.

Given the power-law form of the potential (15), it is useful to note that (cf. Eqs.(9,11)):

$$J(\phi) = \frac{\gamma}{a+1}\phi^{a+1}, \tag{16}$$

$$\phi_o = \left[\frac{a+1}{\gamma}2lGM\right]^{\frac{1}{a+1}}. \tag{17}$$

We can also re-express the black hole mass (10) and temperature (12) as follows:

$$M = \frac{1}{2lG}\frac{\gamma}{a+1}\phi_o^{a+1}, \tag{18}$$

$$T_o = \frac{\gamma}{4\pi l}\phi_o^a, \tag{19}$$

whereas the entropy only depends implicitly (through ϕ_o) on the specific choice of model.

3 Hamiltonian Partition Function

In this section, we will consider the Lorentzian partition function for a 1+1-dimensional (dilatonic) black hole. Formally speaking, this partition function can be defined as follows:

$$\mathcal{Z}[\beta, \phi_+] = Tr\left[\exp(-\beta \hat{H})\right]. \tag{20}$$

Here, ϕ_+ is the fixed value of the dilaton at a timelike outer boundary of the system, β^{-1} is the equilibrium temperature of the system (as measured at the outer boundary), $\hat{H}$ is the Lorentzian Hamiltonian operator (discussed below), and the trace is over all physically relevant states.

The premise of the above formulation is that a black hole can be viewed as thermodynamic object in a heat bath of fixed temperature β^{-1} [25]. It is implied that the entire system has been enclosed in a thermally reflective

[5]This complication can be viewed as a manifestation of such theories admitting black holes with a negative specific heat. The very same problem, of course, notoriously afflicts the Schwarzschild black hole.

"box"; thus ensuring that equilibrium is maintained.[6] The above partition function then appropriately describes the thermodynamics of a canonical ensemble [52].

The ultimate form of the Hamiltonian operator, $\hat{H}$, will depend on how one chooses to define and then foliate the accessible spacetime [21]. For example, previously in a 1+1-dimensional context [41], this author and Kunstatter chose boundary conditions that generalized the spherically symmetric (four-dimensional) formalism of Louko and Whiting [20]. Technically speaking, the spacetime was foliated, from the black hole horizon to the surface of the box, into spacelike hypersurfaces. Moreover, each of these spatial slices was constrained to approach the bifurcation point (i.e., the horizon point of vanishing Killing vector) along a static slice. These boundary conditions are a natural choice from a thermodynamic perspective, inasmuch as the resulting spacetime can be analytically continued to the Euclidean (black hole) instanton [25].

With these specified conditions, the Lorentzian Hamiltonian operator, $\hat{H}_{ex}$,[7] was found to take on the following form [41]:

$$\hat{H}_{ex} = \hat{H}_{+} - \hat{H}_{-}, \tag{21}$$

where:

$$\hat{H}_{+} = \frac{\sqrt{-g^{+}_{tt}J(\phi_{+})}}{lG}\left[1 - \sqrt{1 - \frac{2lG\hat{M}}{J(\phi_{+})}}\right], \tag{22}$$

$$\hat{H}_{-} = \frac{N}{G}\phi_{o}(\hat{M}) = \frac{N}{G}J^{-1}(2lG\hat{M}). \tag{23}$$

Here, $\hat{H}_{\pm}$ is the contribution to the Hamiltonian from the outer/inner boundary (i.e., box surface/horizon), g^{+}_{tt} is the time-time component of the metric at the outer boundary, $\hat{M}$ corresponds to the mass operator, and N is defined according to:

$$N = \frac{d}{dx}\left[\sqrt{\frac{-g_{tt}}{g_{xx}}}\right]\Bigg|_{\phi=\phi_{o}}. \tag{24}$$

[6]Eventually, we consider the asymptotic limit of an infinitely large box or $\phi_{+} \rightarrow \infty$.

[7]Following [35], we label this Hamiltonian with the subscript *ex* for reasons that will soon be made clear.

Taking the limit of an infinitely sized box (i.e., $\phi_+ \to \infty$)[8] and fixing the time coordinate of a boundary observer so that $|g^+_{tt}/J(\phi_+)| \to 1$ at infinity, we obtain the anticipated result:

$$\hat{H}_+ \to \hat{M}. \tag{25}$$

Here, we have chosen a gauge that is quite natural, as it fixes the coordinate time equal to the proper time for an asymptotic observer whose "physical" metric is an asymptotically flat one. Meanwhile, Euclidean thermodynamic considerations can be used to fix the value of N such that $N = 2\pi/\beta$ [41]. Therefore, $H_- = 2\pi\phi_o/G\beta = S_o/\beta$ (cf. Eq.(13)), and so:

$$\hat{H}_{ex} \to \hat{M} - \frac{S_o}{\beta}, \tag{26}$$

which is precisely the (Helmholtz) free energy of the system.

We have included the label *ex* on the Hamiltonian to express that only the *exterior* region of the black hole spacetime (i.e., the region outside of the horizon) has been considered with the above choice of boundary conditions. Such a restriction is known, at least for spherically symmetric gravity, to induce a free-energy form for the Hamiltonian [21], in agreement with our above (and prior [41]) findings.

Alternatively, one could have considered a foliation that covers the entire black hole spacetime, from left-hand-side to right-hand-side asymptotic infinity (including the interior regions), by considering the maximally extended "Kruskal" diagram.[9] It is also of interest to know what the form of the Hamiltonian would be for this latter choice of boundary conditions. This form can, in fact, be readily extrapolated from analogous calculations in the context of spherically symmetric gravity [19, 21, 35]. Fixing the time coordinate at the right boundary as above and freezing time evolution at the left boundary,[10] we have in the asymptotic limit:

$$\hat{H}_{wh} \to \hat{M}, \tag{27}$$

[8] Note that $J(\phi)$ is expected to increase monotonically with ϕ for virtually any model that admits black hole solutions [46].

[9] In our case, this implies a suitable generalization of Kruskal coordinates [53] for 1+1-dimensional gravity. Such coordinates have, indeed, been explicitly formalized in [54].

[10] The physical justification being that an observer should only be capable of making observations at one asymptotic boundary.

where the subscript wh indicates that the *whole* black hole spacetime has now been foliated. Evidently, with this choice of boundary conditions, the Hamiltonian corresponds to the internal energy of the system.

On an intuitive level, one can understand the apparent discrepancy between Eq.(26) and Eq.(27). If the horizon is to be taken seriously, as a causal barrier for a fiducial observer,[11] then such an observer must necessarily be deprived of information about the black hole interior. It is a general principle that any loss of information will directly translate into a gain in entropy. (In the current discussion, this would be an entanglement entropy from a quantum mechanical viewpoint.) This effect is naturally manifested in our formalism via the entropic contribution, $\hat{H}_-$, to the exterior Hamiltonian. Conversely, a hypothetical observer with access to the entire spacetime will assign no special meaning to the horizon and will, therefore, be impervious to its information-negating effects. That is to say, for this privileged observer, $\hat{H}_-$ must effectively vanish.

Recalling that our current objective is to calculate the partition function (20), we must somehow deal with the bothersome issue of having (at least) one Hamiltonian too many. On this point, we will argue that, with proper handling, the partition function describes the same thermodynamics, regardless of whether one chooses to work with $\hat{H}_{ex}$ or $\hat{H}_{wh}$. (As would naturally be expected from any proponent of black hole complementarity [24].) To support this claim, we will follow the logistic framework of Makela and Repo [35]; which is based, in large part, on Bekenstein's notion of black hole spectroscopy [23].

We begin here by considering, in view of the arguments of Bekenstein and others [23, 55], a discrete spectrum for the mass operator of the black hole. Quantitatively speaking, one might expect the following eigenvalue equation:

$$\hat{M}|M_n >= M_n|M_n >, \qquad n = 0, 1, 2, ..., \tag{28}$$

where the mass eigenvalues, M_n, increase monotonically with the quantum number n. We will also assume a level-dependent degeneracy, which will be denoted by $g_{ex}(M_n)$ or $g_{wh}(M_n)$.

Let us concentrate, for the moment, on $\hat{H}_{wh} = \hat{M}$. Directly incorporating this Hamiltonian and the above formalism, we can re-express the partition

[11] That is, a static observer that remains outside of the horizon.

function (20) as follows:

$$\begin{aligned} \mathcal{Z}_{wh}[\beta] &= \sum_{n=0}^{\infty} < M_n | g_{wh}(\hat{M}) e^{-\beta \hat{M}} | M_n > \\ &= \sum_{n=0}^{\infty} g_{wh}(M_n) e^{-\beta M_n}. \end{aligned} \tag{29}$$

We require, at this point, some explicit form of the degeneracy function. From the perspective of statistical mechanics, it is most natural to assume that $\ln g_{wh}(M_n) = S_o(M_n)$, where S_o is the classical black hole entropy (13). Also employing Eq.(11), we thus have:

$$\mathcal{Z}_{wh}[\beta] = \sum_{n=0}^{\infty} \exp\left(-\beta M_n + \frac{2\pi}{G} J^{-1}(2lGM_n)\right). \tag{30}$$

Let us next consider $\hat{H}_{ex}$ as given by Eqs.(21,23,25). With this choice of Hamiltonian, the partition function (20) now translates into:

$$\mathcal{Z}_{ex}[\beta] = \sum_{n=0}^{\infty} g_{ex}(M_n) \exp\left[-\beta\left(M_n - \frac{N}{G} J^{-1}(2lGM_n)\right)\right]. \tag{31}$$

What about the degeneracy function in this case? Here, we will argue that $g_{ex}(M_n)$ should be unity (for all n) on the grounds of the black hole no-hair theorem [56]. That is to say, an asymptotic observer who is restricted to the exterior should know nothing about the internal degrees of freedom of the black hole (which are presumably encoded within the degeneracy). Rather, such an observer should only be able to account for the macroscopic parameters; which, in this toy model, are exclusively restricted to the mass.

Setting $g_{ex}(M_n) = 1$ and also fixing $N = 2\pi/\beta$ (as follows from Euclidean thermodynamic considerations [41]), we find that $\mathcal{Z}_{wh}$ and $\mathcal{Z}_{ex}$ are indeed indistinguishable. Thus, we are lead to conjecture a "universal" partition function of the following form:

$$\mathcal{Z}[\beta] = \sum_{n=0}^{\infty} \exp\left(-\beta M_n + \frac{2\pi}{G} J^{-1}(2lGM_n)\right). \tag{32}$$

A point of interest: if one would rather avoid the use of Euclidean thermodynamics to fix N (which may seem unsavory in view of our otherwise

Lorentzian framework), then black hole complementarity [24] might have alternatively been employed for this very purpose. To put it another way, if one subscribes *a priori* to observer-independent physics, then N can be constrained (by identifying the partition functions) independently of any other considerations.

4 Quantum-Corrected Entropy

With a single, well-defined expression for the partition function (32), we are now in a position for a quantum calculation of the black hole entropy. If we are going to proceed, however, an explicit form for the mass eigenvalues, M_n, will first be required. To resolve this matter, let us first take note of Bekenstein's proposal [23] for the eigenvalues of the horizon area:[12]

$$A_n = \epsilon n l_p^2, \qquad n = 0, 1, 2, ..., \tag{33}$$

where ϵ is a numerical constant of order unity and l_p is the Planck length. This expression, of course, strictly applies to a four-dimensional theory. Nonetheless, in view of Eqs.(14,11), the most obvious two-dimensional analogue can be written as:

$$J^{-1}(2lGM) = \epsilon n, \qquad n = 0, 1, 2, ..., \tag{34}$$

or perhaps more usefully as:

$$M_n = \frac{1}{2lG} J(\epsilon n), \qquad n = 0, 1, 2, ..., \tag{35}$$

where ϵ is a constant of order G.

For definiteness, let us now call upon the power-law model as stipulated by Eqs.(15,16). In this case:

$$M_n = \frac{\gamma}{a+1} \frac{(\epsilon n)^{a+1}}{2lG}. \tag{36}$$

[12] In later adaptations (for instance, [55]), there is often a zero-point term included as well. However, for a sufficiently massive black hole, such a term is essentially irrelevant and can be safely neglected.

Substituting the above result into Eq.(32), we have for the (universal) partition function:

$$\mathcal{Z}[\beta] = \sum_{n=0}^{\infty} \exp\left(-\beta\frac{\gamma}{a+1}\frac{(\epsilon n)^{a+1}}{2lG} + \frac{2\pi}{G}\epsilon n\right). \tag{37}$$

It is quite evident that this summation converges, provided that $a \geq 0$, as has been priorly assumed. However, the case of $a < 0$ could still be readily handled via Makela and Repo's approach [35]; that is, calculating the partition function for the black hole radiation rather than the black hole *per se*. (This maneuver effectively reverses the sign of the exponent.) In this way, one can verify that our final outcome (for the entropy) can be extended to any choice of a.

Let us now assume that the black hole is sufficiently massive so that the summation can be replaced by an integral:

$$\mathcal{Z}[\beta] = \int_0^{\infty} dy \exp\left(-\beta\frac{\gamma}{a+1}\frac{y^{a+1}}{2lG} + \frac{2\pi}{G}y\right), \tag{38}$$

where an obvious change has been made in the integration variable.

Without resorting to numerics, one can still proceed by employing a saddle-point type of approximation. First of all, a straightforward calculation reveals an extremum in the exponent at $y = y_o$, where $y_o^a = 4\pi l/\beta\gamma$. Next, we can expand about this extremal point to obtain:

$$\mathcal{Z}[\beta] \approx e^{\frac{2\pi a}{G(a+1)}\left[\frac{4\pi l}{\beta\gamma}\right]^{\frac{1}{a}}} \int_0^{\infty} dy \exp\left(-\frac{a\pi}{\gamma}\left[\frac{\beta\gamma}{4\pi l}\right]^{\frac{1}{a}}(y-y_o)^2\right). \tag{39}$$

This expression is readily integrated to yield:

$$\mathcal{Z}[\beta] \approx e^{\frac{2\pi a}{G(a+1)}\left[\frac{4\pi l}{\beta\gamma}\right]^{\frac{1}{a}}} \sqrt{\frac{\gamma}{4a}}\left(\frac{4\pi l}{\beta}\right)^{\frac{1}{2a}}\left[1 + erf(\sqrt{\omega})\right], \tag{40}$$

where:

$$\omega = \frac{a\pi}{\gamma}\left(\frac{4\pi l}{\beta\gamma}\right)^{\frac{1}{a}} \tag{41}$$

and erf denotes the error function.[13]

[13] To be precise: $erf(\alpha) = 2\int_0^{\alpha} d\xi \exp(-\xi^2)/\sqrt{\pi}$.

As implied above, we are specifically focusing on a semi-classical regime; that is, the black hole mass should be regarded as large relative to any fundamental scale (which is a typical convention in studies of this nature). It is clear from Eqs.(18,19) that, for the model of interest, the temperature increases monotonically with mass. Hence, this semi-classical regime can be equally well portrayed as one of high temperature. This means that, under current considerations, we are justified in regarding β as small in units of the fundamental length scale, l. For this reason, ω can be viewed as a very large (dimensionless) quantity; thus implying that $erf(\sqrt{\omega}) \approx 1$ (up to corrections that vanish in the limit of infinite temperature).

Let us consider the standard formula for extracting entropy from a thermodynamic partition function:

$$S = \left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln\mathcal{Z}. \tag{42}$$

Substituting for the partition function (40) and keeping the semi-classical approximation in mind, we eventually have:

$$S = \frac{2\pi}{G}\left(\frac{4\pi l}{\beta\gamma}\right)^{\frac{1}{a}} + \frac{1}{2a}\ln\left(\frac{l}{\beta}\right) + \mathcal{O}\left[\left(\frac{\beta}{l}\right)^{\frac{1}{a}}\right] + constant. \tag{43}$$

Let us remind ourselves that β^{-1} is defined as the equilibrium temperature measured by an asymptotic observer. This is simply the definition of the Hawking temperature,[14] and so it follows that $\beta^{-1} = T_o$. With this identification, we can apply Eq.(19) for T_o, along with Eq.(13) for the classical entropy (S_o), and re-express Eq.(43) as follows:

$$S = S_o + \frac{1}{2}\ln(S_o) + \mathcal{O}\left[S_o^{-1}\right] + constant. \tag{44}$$

It is a nice consistency check that we have reproduced the classical thermodynamic value with the lowest-order term. (Even if this was, more or less, pre-ordained by some assumptions made in Section 3.) A substantially

[14] Recall from Section 3 that the time coordinate of the boundary observer has been fixed so that the "physical" metric is asymptotically flat. Therefore, as far as an asymptotic observer is concerned, the Tolman red-shift factor is equal to unity. Hence, the asymptotic boundary temperature does indeed coincide with the Hawking value.

less trivial outcome is, however, the first-order quantum correction. Here, we have confirmed the expected logarithmic dependence and, moreover, found a prefactor of $+1/2$. It is interesting and perhaps disturbing that this prefactor disagrees, in both sign and magnitude, with the values obtained (for the very same model) in prior calculations [26]. We will have more to say on this discrepancy in the concluding section.

5 Conclusion

In summary, we have been investigating a generic theory of 1+1-dimensional dilaton-gravity; with special attention on models with a power-law form of potential. After some preliminaries, we employed a Lorentzian Hamiltonian approach as a means for studying the thermodynamics of two-dimensional black holes. Black hole spectroscopy also played a substantial role in establishing a calculable expression for the quantum partition function. Ultimately, we were able to derive an expression for the entropy as an expansion in diminishing powers of Hawking temperature. (This temperature can be regarded as large in a semi-classical regime of massive black holes.) Reassuringly, the tree-level term was shown to be in precise agreement with the classical thermodynamic entropy. Moreover, we have substantiated that the leading-order correction is directly proportional to the logarithm of the classical entropy. However, the prefactor for this logarithmic correction, $+1/2$, is in conspicuous disagreement with the values obtained (for the exact same model) by other methodologies [26].

An interesting feature of our formalism (specifically in Section 3) is that it provides a simple but powerful framework for understanding how black hole complementarity [24] can resolve the information loss paradox [5]. As far as a fiducial observer is concerned, the black hole horizon is a physical, causal boundary behind which information can indeed be lost. This apparent loss of information manifests itself as an entanglement entropy (between the interior and exterior regions) and is directly evident as an entropic contribution to the free energy (cf. Eq.(26)). Conversely, from a global perspective, the horizon is merely an artifact of poorly chosen coordinates, and there can be no loss of information in this frame of reference. Rather, the black hole entropy arises from an inherent degeneracy in the discrete levels of the mass or area spectrum. (Meanwhile, the fiducial observer has no knowledge of this degeneracy by virtue of the no-hair theorem.) That these two viewpoints have turned out to be thermodynamically equivalent can be viewed as a manifestation of black hole complementarity. (For further discussion, see [35].)

Finally, let us more closely examine the apparent discrepancy that has arisen at the logarithmic order in the entropy. To review [26], a thermodynamic calculation (based on [27]) yields a prefactor of $-\frac{1}{2}(2a+1)$, a statistical approach (based on [28]-[30]) gives a value of $-3/2$, while the current, Hamiltonian program generates a value of $+1/2$.

Assuming the credibility of all three methodologies, one is inclined to take an inventory of any possible limitations. In this regard, it is perhaps relevant that the statistical approach is based on a near-horizon duality and only accounts for degrees of freedom within the vicinity of the horizon. However, there is strong support for the notion that this is precisely where the degrees of freedom of a black hole spacetime should reside (for instance, [57]). On the other hand, we should re-emphasize that *both* the thermodynamic and Hamiltonian method invoke an assumption of large temperature. For the two-dimensional model of interest, this regime of high temperature can best be viewed as a semi-classical approximation. Since we are considering a *quantum* correction to the entropy, it is quite possible that this regime is too restrictive. On this basis, one might argue that, in a true quantum analysis, the thermodynamic and Hamiltonian prefactors will both converge towards the statistical value of $-3/2$.

Further support that the statistical value of the prefactor is, in some sense, the fundamental one follows from calculations in a quantum-geometry context. That is to say, a viable candidate for a quantum theory of gravity also yields a value of $-3/2$ [31]. It may also be pertinent that the statistical approach is based directly on the Cardy formula [29]. More to the point, it has been argued that, exclusively for a two-dimensional theory, the Cardy formula counts the true microstates of the black hole when it is used in this manner [58]. If this is literally correct, the statistical calculation would indeed be entitled to an elevated status.

It would be interesting if the above arguments could be supported (or disputed) in a more rigorous way, and we hope to address this matter in a future study.

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